Journal of Statistical Physics, Vol. 21, No. 3, 1979

Tunneling and Transport Problems for a Quantum Mechanical Brownian Particle

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Received July 27, 1978; revised October 24, 1978, and January 26, 1979

The Kanai model of a quantum mechanical Brownian particle is used to examine the effect of interactions between particles and their environment. Random forces cause the thermalization of the particle. Reflection of a particle from a step barrier is analyzed. The problem of tunneling of the Brownian particle through a rectangular barrier is solved. Finally, a solution for a Brownian particle in a box is presented.

KEY WORDS: Quantum mechanical friction; tunneling; Brownian motion; transport; energy dissipation.

1. INTRODUCTION

In 1949 Kanai[(1)] introduced a quantum mechanical Hamiltonian intended for disspative systems. Since then a number of papers have examined his theory in some detail and other dissipative Hamiltonians have been invented (see Hasse[(2)] for a review of this work).

The advantage of constructing such Hamiltonians is that they allow the writing of a wave equation for a particle interacting frictionally with its environment. As with the classical Brownian particle, the interaction is modeled by a single parameter ζ, the friction constant, which is a manifestation of the randomly fluctuating forces representing the environmental particles. The alternative to this approach is to develop a many-body Hamiltonian and to extract information from the conventional Schrödinger equation via perturbation analysis. Our present objective is more limited, in that we focus our attention on one particle in a bath of background particles. The particle is quantum mechanical with no internal degrees of freedom; the bath is classical. The particle responds to environmental random forces based on Boltzmann statistics. In this paper we show how the Kanai theory

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0022-4715/79/0900-0301$03.00/0

for the motion of a single particle in a dissipative background provides insight into tunneling and transport phenomena. The examples are simple enough to be solved analytically in closed form, but extensions to more complicated situations are possible.

2. WAVE MECHANICS OF A BROWNIAN PARTICLE

The Kanai quantum mechanical Hamiltonian operator can be written[1]

$$\hat{H} = (-\hbar^2/2m)e^{-\gamma t}(\partial^2/\partial x^2) + [V_E(x) + V_R(x, t)]e^{\gamma t} \tag{1}$$

in terms of $\gamma = \zeta/m$, the ratio of the dynamical friction constant to the mass of the particle; $V_E(x)$, an external potential; and $V_R(x, t)$, the potential of the randomly fluctuating force characteristic of Brownian motion. The external and random forces are given, respectively, by

$$F_E(x) = -\partial V_E(x)/\partial x \tag{2}$$

and

$$F_R(t) = -\partial V_R(x, t)/\partial x \tag{3}$$

The one-dimensional wave equation becomes

$$i\hbar\, \partial\psi/\partial t = (-\hbar^2/2m)e^{-\gamma t}\, \partial^2\psi/\partial x^2 + e^{\gamma t} V(x, t)\psi(x, t) \tag{4}$$

where

$$V(x, t) = V_E(x) + V_R(x, t) \tag{5}$$

In the limit $\gamma \to 0$, Eq. (4) reduces to the conventional Schrödinger equation.

It is of interest to investigate how the rate of loss of energy of the system to the surroundings is related to the friction coefficient γ. For this purpose, consider the Langevin equation of motion for Heisenberg operators [2]

$$d\hat{p}/dt = -\gamma\hat{p} + F_E(\hat{x}) + F_R(t) \tag{6}$$

As in the classical Brownian motion problem, the generalized momentum operator is different by a factor of $e^{\gamma t}$ from the Heisenberg momentum operator $\hat{p}$.

The solution to Eq. (6) for a free Brownian particle ($V_E = 0$) is used to obtain the expectation value and ensemble average of the kinetic energy, which for $\overline{F_R(s)} = 0$ becomes

$$\langle\bar{\hat{K}}\rangle = \langle\overline{\hat{p}^2(0)/2m}\rangle e^{-2\gamma t} + \tfrac{1}{2}kT(1 - e^{-2\gamma t}) \tag{7}$$

where we have considered the bath to be classical so that the correlation function can have the form

$$\overline{F_R(s)F_R(s')} = 2m\gamma kT\, \delta(s - s') \tag{8}$$

Since the ensemble average of the fluctuating potential $V_R(x, t)$ is zero, the total energy of the system is equal to the kinetic energy $\langle \hat{E} \rangle = \langle \hat{K} \rangle$. Then, differentiating Eq. (7) with respect to time gives us a differential equation whose solution is

$$\langle \bar{\hat{E}} \rangle = (\langle \overline{\hat{E}(0)} \rangle - kT/2)e^{-2\gamma t} + kT/2 \tag{9}$$

Thus the relaxation of the energy of an ensemble of free Brownian particles proceeds to a value $kT/2$ with a time constant of $1/2\gamma$ for large times. Therefore the Kanai model of quantum mechanical friction not only shows that the particle loses energy,[3] but shows also how the particle is thermalized.

Haase[2] has presented an extended discussion of solutions to the Schrödinger equation (4) when the random field is absent, i.e., when $V_R(x, t) = 0$. Because of the inseparable relation between the random potential and the friction coefficient in Brownian motion, we will present the solution to Eq. (4) when

$$V(x, t) = -xF(t) \tag{10}$$

where

$$F(t) = F_E(t) + F_R(t) \tag{11}$$

For the translating and damped particle we assume a form for the wave function

$$\psi(x, t) = \exp\{i[x\alpha(t) - \phi(t)]\} \tag{12}$$

where $\alpha(t)$ and $\phi(t)$ are functions of t only. Substituting Eq. (12) into Eq. (4) gives

$$x[\hbar\, d\alpha/dt - e^{\gamma t}F(t)] = \hbar\, d\phi/dt - (\hbar^2\alpha^2/2m)e^{-\gamma t} \tag{13}$$

Since the left-hand side of Eq. (13) is a product of the variable x and a function of t, while the right-hand side is a function of t only, Eq. (13) will be satisfied only if

$$\hbar\, d\alpha(t)/dt = e^{\gamma t}F(t) \tag{14}$$

and

$$d\phi(t)/dt = (\hbar/2m)e^{-\gamma t}\alpha^2(t) \tag{15}$$

Integrating Eqs. (14) and (15) yields expressions for α and ϕ to be used in Eq. (12),

$$\alpha(t) = \alpha_0 + (1/\hbar)\int_0^t e^{\gamma s}F(s)\, ds \tag{16}$$

and

$$\phi(t) = \phi_0 + (\hbar/2m)\int_0^t e^{-\gamma s}\alpha^2(s)\,ds \tag{17}$$

with $\alpha_0 = \alpha(t = 0)$ and $\phi_0 = \phi(t = 0)$. The solution has the property of correspondence,[2] and also reduces as $\gamma \to 0$ to the wave function for a freely translating particle in a conservative system with $\phi(t) = \omega t$. The wave function, Eqs. (12), (16), and (17), for the translating Brownian particle is the starting point for the analyses to follow.

As an example of how the quantum theory of Brownian motion can be used to obtain a realistic physical result, Buch and Denman,[4] by neglecting $F_R(t)$, have derived the expression for conduction flux of charged particles in an electric field. For the random part of the flux it is not difficult to show that

$$\langle j_R{}^2\rangle = \rho^2\langle v_2\rangle(1 - e^{-2\gamma t}) \tag{18}$$

in terms of the particle density ρ and thermal velocity of the charged particles[5]

$$\langle v^2\rangle = kT/m \tag{19}$$

After the transient has passed we have the rms value of the current per unit charge due to the random forces,

$$j_R^{\mathrm{rms}} = \rho(kT/m)^{1/2} \tag{20}$$

3. REFLECTION FROM A BARRIER

Consider the step potential

$$V(x) = \begin{cases} V_0 > 0 & \text{for } x > x_0 \\ 0 & \text{for } x < x_0 \end{cases} \tag{21}$$

We assume that the Schrödinger–Langevin equation (4) has a solution of the form

$$\psi_{\mathrm{I}} = e^{i(\alpha x - \phi)} + Ae^{-i(\alpha' x + \phi')} \qquad \text{for } x < x_0 \tag{22}$$

$$\psi_{\mathrm{II}} = Be^{i(\alpha'' x - \phi'')} \qquad \text{for } x > x_0 \tag{23}$$

where α and $\phi(t)$ are given by Eqs. (16) and (17). To determine expressions for the reflected wave parameters α' and ϕ', we follow the steps used to get α and ϕ, i.e., Eqs. (12)–(17), but with the wave function of Eq. (12) replaced with the term multiplied by A in Eq. (22). One obtains

$$\alpha' = \alpha_0 - \hbar^{-1}\int_0^t e^{\gamma s}F_R(s)\,ds \tag{24}$$

and

$$\phi' = \phi_0 + (\hbar/2m)\int_0^t e^{-\gamma s}[\alpha'(s)]^2\, ds \tag{25}$$

Of course $\phi' \neq \phi$, since $\alpha' \neq \alpha$. For the transmitted wave we require that energy be conserved; therefore,

$$\hbar^2\alpha^2/2m = \hbar^2(\alpha'')^2/2m + V_0 \tag{26}$$

so that

$$\alpha'' = (\alpha^2 - 2mV_0/\hbar^2)^{1/2} \tag{27}$$

Since for $x > x_0$ we have $V_E = V_0$, it follows that

$$\phi'' = \phi_0 + (\hbar/2m)\int_0^t e^{-\gamma s}[\alpha''(s)]^2\, ds + V_0(e^{\gamma t} - 1)/\gamma\hbar \tag{28}$$

To determine the constants A and B we use the condition that ψ and its x derivative both be continuous at $x_0 = 0$; thus

$$e^{-i\phi} + Ae^{-i\phi'} = Be^{-i\phi''} \tag{29}$$

and

$$\alpha e^{-i\phi} - \alpha' A e^{-i\phi'} = \alpha'' B e^{-i\phi''} \tag{30}$$

Solving for A and B yields

$$A = \frac{\alpha - \alpha''}{\alpha' + \alpha''}\, e^{i(\phi' - \phi)} \tag{31}$$

and

$$B = \frac{\alpha' + \alpha}{\alpha' + \alpha''}\, e^{i(\phi'' - \phi)} \tag{32}$$

From Eq. (23) we have

$$\psi_{\mathrm{II}} = \frac{\alpha' + \alpha}{\alpha' + \alpha''}\, e^{-i(\phi - \alpha'' x)} \tag{33}$$

so that ϕ'' is not explicitly required.

The reflectivity is obtained from Eq. (31),

$$|A|^2 = AA^* = \left(\frac{\alpha - \alpha''}{\alpha' + \alpha''}\right)^2 \tag{34}$$

when $\alpha^2 > 2mV_0/\hbar^2$, so that α'', defined in Eq. (27), is real. When $\alpha^2 \leqslant 2mV_0/\hbar^2$, α'' is imaginary and $|A|^2$ equals unity, since the energy of the incoming wave has energy less than the barrier height.[6]

Because Eq. (34) contains contributions from the random force, we need an ensemble average of the reflectivity to assess the effect of frictional damping. From Eqs. (16) and (24) we write

$$\alpha' = \alpha_0 - \lambda/\hbar \tag{35}$$

$$\alpha = \alpha_0 + \lambda/\hbar \tag{36}$$

where

$$\lambda = \int_0^t e^{\gamma s} F_R(s)\, ds \tag{37}$$

A lemma of Chandrasekar[7] provides the probability distribution of λ,

$$W(\lambda) = \left[4\pi q \int_0^t \psi^2(s)\, ds\right]^{1/2} \exp\left[(-\lambda_2/4q)\int_0^t \psi^2(s)\, ds\right] \tag{38}$$

where

$$\psi = e^{\gamma s} \tag{39}$$

and

$$q = \gamma kT/m \tag{40}$$

Therefore, the ensemble average of Eq. (34) may be calculated from

$$\overline{|A|^2} = \int_{-\infty}^{\infty} |A|^2 W(\lambda)\, d\lambda \tag{41}$$

A more restricted result may be obtained for $\gamma t \ll 1$, so that λ is assumed to be a very small quantity in the sense that $\overline{\lambda^2}$ is small. We use the power series expansion and keep terms up to λ^2 to show that Eq. (27) can be written

$$\alpha'' \approx \beta[1 - \lambda\alpha_0/\hbar\beta^2 + \lambda^2(\tfrac{1}{2}\hbar^{-2}\beta^{-2})(1 - \alpha_0{}^2/\beta^2)] \tag{42}$$

with

$$\beta^2 = \alpha_0{}^2 - 2mV_0/\hbar^2 \tag{43}$$

not equal to zero in this expansion. Next we substitute Eq. (42) into (34), expand the reciprocal of the denominator, and keep terms up to λ^2. We take the ensemble average of the resulting expression and, recognizing that $\bar{\lambda} = 0$ since $\overline{F_R} = 0$, we obtain

$$\overline{|A|^2} \approx \left(\frac{\alpha_0 - \beta}{\alpha_0 + \beta}\right)^2 \left\{1 + \overline{\lambda^2}\,\frac{2}{\beta^2\hbar^2}\left[1 + \frac{2\alpha_0(2\alpha_0 - \beta)}{(\alpha_0 + \beta)^2}\right]\right\} \tag{44}$$

where

$$\overline{\lambda^2} = \int_0^t ds \int_0^t ds'\, e^{\gamma(s+s')}\overline{F_R(s)F_R(s')} \tag{45}$$

is given in terms of the correlation function of the random force. According to Eq. (44), the effect of frictional damping is to increase the reflectivity above its value $(\alpha_0 - \beta)^2/(\alpha_0 + \beta)^2$ in the absence of friction.

Since the correlation function in (45) is equivalent to the one in Eq. (7), we can write

$$\overline{\lambda^2} = mkT(e^{2\gamma t} - 1) \approx 2\gamma tmkT \tag{46}$$

since γt must be small to ensure that γ is small. Thus reflectivity increases and tunneling decreases as T, m, or γt increases.

4. TUNNELING THROUGH A RECTANGULAR BARRIER

Consider the rectangular potential

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0 > 0, & x < x < l \\ 0, & x > l \end{cases} \tag{47}$$

We assume that the wave function has the following forms in the three regions:

$$x < 0; \qquad \psi_{\mathrm{I}} = e^{i(\alpha x - \phi)} + Ae^{-i(\alpha_1 x + \phi_1)} \tag{48}$$

$$0 < x < l: \qquad \psi_{\mathrm{II}} = Be^{i(\alpha_2 x - \phi_2)} + Ce^{-i(\alpha_3 x + \phi_3)} \tag{49}$$

$$x > l: \qquad \psi_{\mathrm{III}} = De^{i(\alpha_4 x - \phi_4)} \tag{50}$$

in terms of

$$\alpha = \alpha_0 + \lambda/\hbar \tag{51}$$

$$\alpha_1 = \alpha_0 - \lambda/\hbar \tag{52}$$

$$\alpha_2 = (\alpha^2 - 2mV_0/\hbar^2)^{1/2} \tag{53}$$

$$\alpha_3 = (\alpha_1{}^2 - 2mV_0/\hbar^2)^{1/2} \tag{54}$$

$$\alpha_4 = (\alpha_2{}^2 + 2mV_0/\hbar^2)^{1/2} = \alpha \tag{55}$$

where Eqs. (53)–(55) are results of conservation of energy as in Eq. (27). If $\gamma = 0$ and $F_R(t) = 0$, then $\alpha = \alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$ as for the undamped wave.[6] The quantity ϕ is given by Eq. (17), and because $\alpha = \alpha_4$ in Eq. (55), it follows that $\phi_4 = \phi$. Although equations for ϕ_1, ϕ_2, and ϕ_4 are readily derived using the same procedure as for Eqs. (17) and (25), these quantities cancel from our final expressions and therefore do not appear explicitly.

Applying conditions that ψ and its x derivative are continuous at $x = 0$ and $x = l$ yields

$$e^{-i\phi} + Ae^{-i\phi_1} = Be^{-i\phi_2} + Ce^{-i\phi_3} \tag{56}$$

$$\alpha e^{-i\phi} - \alpha_1 Ae^{-i\phi_1} = \alpha_2 Be^{-i\phi_2} - \alpha_3 Ce^{-i\phi_3} \tag{57}$$

$$Be^{i(\alpha_2 l - \phi_2)} + Ce^{-i(\alpha_3 l + \phi_3)} = De^{i(\alpha_4 l - \phi_4)} \tag{58}$$

$$\alpha_2 Be^{i(\alpha_2 l - \phi_2)} - \alpha_3 Ce^{-i(\alpha_3 l + \phi_3)} = \alpha_4 De^{i(\alpha_4 l - \phi_4)} \tag{59}$$

We will solve these equations to obtain $|D|^2$, the transmission coefficient of the barrier.

If Eq. (58) is divided by (59), we obtain an equation for B in terms of C. Substituting this equation into Eqs. (56) and (57) and dividing gives an equation for A,

$$A = \frac{\alpha\Gamma_1 + \Gamma_2}{\alpha_1\Gamma_1 + \Gamma_2} e^{-i(\phi - \phi_1)} \tag{60}$$

where

$$\Gamma_1 = \frac{\alpha_3 + \alpha_4}{\alpha_2 - \alpha_4} e^{-i[l(\alpha_3 + \alpha_2) + \phi_3]} + e^{-i\phi_3} \tag{61}$$

$$\Gamma_2 = \alpha_2 \frac{\alpha_3 + \alpha_4}{\alpha_2 - \alpha_4} e^{-i[l(\alpha_3 + \alpha) + \phi_2 + \phi_3 - \phi_4]} - \alpha_3 e^{-i\phi_3} \tag{62}$$

We also find

$$C = \frac{2\alpha_1 e^{-i\phi}}{\alpha_1\Gamma_1 + \Gamma_2} \tag{63}$$

$$B = \frac{\alpha_3 + \alpha_4}{\alpha_2 - \alpha_4} \frac{2\alpha_1}{\alpha_1\Gamma_1 + \Gamma_2} e^{-i[l(\alpha_2 + \alpha_3) + \phi_3 - \phi_2 + \phi]} \tag{64}$$

$$D = \frac{\alpha_2 + \alpha_3}{\alpha_2 - \alpha_4} \frac{2\alpha_1}{\alpha_1\Gamma_1 + \Gamma_2} e^{-i[l(\alpha + \alpha_3) + \phi + \phi_3 - \phi_4]} \tag{65}$$

In the limit where $\gamma = 0$ and $F_R = 0$, it is not difficult to show that the equations for A and D are equivalent to ones given by Schiff.[6]

For α^2, $\alpha_1^2 < 2mV_0/\hbar^2$ we can show after some algebraic manipulation that the transmission coefficient for the barrier for small λ is

$$\overline{|D|^2} = \left[1 + \frac{\sinh^2\eta}{4\epsilon(1-\epsilon)}\right]^{-1}\left\{1 + \gamma'\left[1 - \frac{\epsilon}{(1-\epsilon)^2}\frac{\epsilon\eta}{1-\epsilon} + \Theta\right]\right\} \tag{66}$$

where

$$\Theta = \left[1 + \frac{\sinh^2 \eta}{4\epsilon(1-\epsilon)}\right]^{-1} \left\{ \frac{\epsilon(2\epsilon - 1)}{(1-\epsilon)^2} \sinh^2 \eta - \left[\frac{\eta}{4}\frac{(2\epsilon-1)^2}{(1-\epsilon)^3} + \frac{\eta\epsilon}{(1-\epsilon)^2}\right] \right.$$

$$\left. \times \sinh \eta \cosh \eta + \frac{1}{4}\frac{\epsilon}{1-\epsilon} \cosh \eta \left[e^{-\eta} - 3e^{\eta} - \frac{2\epsilon}{1-\epsilon} \cosh \eta \right] \right\} \tag{67}$$

In these equations we have used Eq. (46) and defined

$$\eta = [2\nu(1-\epsilon)]^{1/2} \tag{68}$$

with

$$\nu = mV_0 l^2/\hbar^2 \tag{69}$$

$$\epsilon = \alpha_0{}^2\hbar^2/2mV_0 \tag{70}$$

and

$$\gamma' = 2m\gamma tkT/\alpha_0{}^2\hbar^2 \tag{71}$$

As $\gamma \to 0$ we recover the transmission coefficient for the undamped wave. Of course it is also possible to write an ensemble average of the transmission coefficient with the probability distribution of λ given by Eq. (38); that is,

$$\overline{|D|^2} = \int_{-\infty}^{\infty} |D|^2 W(\lambda)\, d\lambda \tag{72}$$

5. DAMPED PARTICLE IN A BOX

Consider the square well potential with impenetrable walls,

$$V(x) = \begin{cases} 0, & 0 < x < l \\ \infty, & x < 0 \text{ and } x > l \end{cases} \tag{73}$$

We assume a form for the wave function

$$\psi(x, t) = Ae^{i(\alpha x - \phi)} + Be^{-i(\alpha' x + \phi')} \tag{74}$$

for waves traveling in different directions. The parameters can be written as

$$\alpha = \alpha_0 + \lambda/\hbar \tag{75}$$

$$\alpha' = \alpha_0' - \lambda/\hbar \tag{76}$$

$$\phi = \phi_0 + \frac{\hbar}{2m}\int_0^t e^{-\gamma s}[\alpha'(s)]^2\, ds \tag{77}$$

$$\phi' = \phi_0' + \frac{\hbar}{2m}\int_0^t e^{-\gamma s}[\alpha'(s)]^2\, ds \tag{78}$$

We require that the wave function vanish where the potential is infinite; thus

$$0 = \psi(x = 0) = Ae^{-i\phi} + Be^{-i\phi'} \tag{79}$$

and

$$0 = \psi(x = l) = Ae^{i(\alpha l - \phi)} + Be^{-i(\alpha' l + \phi')} \tag{80}$$

From Eq. (79) we find

$$A/B = -e^{-i(\phi' - \phi)} \tag{81}$$

Substituting Eq. (81) into (80) and simplifying yields

$$-\exp(i\alpha_0 l) + \exp(-i\alpha_0' l) = 0 \tag{82}$$

But for our problem to reduce to the conventional undamped wave case, we must have

$$\alpha_0 = \alpha_0' \tag{83}$$

Thus Eq. (82) specifies

$$\sin(\alpha_0 l) = 0 \tag{84}$$

which requires that

$$\alpha_0 = n\pi/l \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{85}$$

These are the same eigenvalues as for the undamped case.

The wave function can be written as

$$\psi = -2Bi \exp[i(\lambda x/\hbar - \phi') \sin \alpha_0 x] \tag{86}$$

which has the same normalization as the undamped case:

$$1 = \int_0^l \psi^* \psi \, dx \tag{87}$$

from which

$$|B|^2 = 1/(2l) \tag{88}$$

We next examine momentum and energy of the particle described by the wave function (86).

The expectation value of the generalized momentum is

$$\langle \hat{P} \rangle = -i\hbar \int_0^l \psi^* \frac{\partial \psi}{\partial x} \, dx \tag{89}$$

With (88) one finds

$$\langle \hat{P} \rangle = \lambda = \int_0^t e^{\gamma s} F_R(s) \, ds \tag{90}$$

This result is exactly what one obtains for the average momentum of the two reflected waves; i.e.,

$$\tfrac{1}{2}[\hbar\alpha - \hbar\alpha'] = \lambda \tag{91}$$

The observable momentum is given by[2]

$$\langle \hat{p} \rangle = \langle \hat{P} \rangle e^{-\gamma t} = \lambda e^{-\gamma t} \tag{92}$$

so that friction eventually causes the momentum of the particle to vanish.

The expectation value of the Hamiltonian is

$$\langle \hat{H} \rangle = \int_0^l \psi^* \hat{H} \psi \, dx \tag{93}$$

where $\hat{H}$ is given by Eq. (1) with $V_E = 0$. Performing the indicated operations yields

$$\langle \hat{H} \rangle = (1/2m)(\lambda^2 + \hbar^2\alpha_0{}^2)e^{-\gamma t} + e^{\gamma t}\langle V_R(x, t)\rangle \tag{94}$$

Using the relation between the total energy of the system and the Hamiltonian yields

$$\langle \hat{E} \rangle = \langle \hat{H} \rangle e^{-\gamma t} = (1/2m)(\lambda^2 + \hbar^2\alpha_0{}^2)e^{-2t} + \langle V_R(x, t)\rangle \tag{95}$$

We perform an ensemble average on Eq. (95) and use $\overline{F_R(t)} = 0$, along with the expression for $\overline{\lambda^2}$, which appears in Eq. (46), to obtain

$$\overline{\langle \hat{E} \rangle} = (\hbar^2/2m)\alpha_0{}^2 e^{-2\gamma t} + \tfrac{1}{2}kT(1 - e^{-2\gamma t}) \tag{96}$$

which is essentially equivalent to Eq. (9). As $t \to \infty$, the system is thermalized to energy $kT/2$.

In summary, we have applied the Kanai theory of quantum mechanical friction to several fundamental problems of a particle interacting with its environment. The random forces due to the environment give rise to the dissipative force proportional to particle velocity in the manner of the usual Langevin drag, and cause the eventual thermalization of the particle. This damping effect influences the behavior of a particle reflecting from a step barrier or tunneling through a rectangular barrier and of the particle in a box.

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